Dynamical model of a cooperative driving system for freeway traffic

Katsuya Hasebe*
Faculty of Business Administration, Aichi University, Miyoshi, Aichi 470-0296, Japan

Akihiro Nakayama†
Gifu Keizai University, Ohgaki, Gifu 503-8550, Japan

Yūki Sugiyama‡
Graduate School of Information Science, Nagoya University, Nagoya 464-8601, Japan

(Received 26 December 2002; revised manuscript received 16 May 2003; published 5 August 2003)

We propose an extended optimal velocity model applicable to cooperative driving control system, which will be realized in the near future. In the model, a vehicle is controlled by the system using the information of arbitrary number of vehicles that precede or follow. We investigate the stability of uniform flow and the response to a disturbance in the linear approximation.

DOI: 10.1103/PhysRevE.68.026102 PACS number(s): 05.70.Fh, 45.70.Vn, 07.05.Tp, 07.05.Dz

I. INTRODUCTION

On highways, one can find that free traffic flow changes to congested flow as the vehicle density increases. To explain this phenomenon, a lot of studies have been done from the physical viewpoint [1–5]. There have been many attempts constructing models for traffic flow: cellular automaton models [6–8], fluid dynamical models [9], coupled map models [10], and a probabilistic model using the master equation [11]. We proposed a dynamical model of traffic flow, an optimal velocity (OV) model [12], which is one of car following models. These models have successfully described the dynamical formation of traffic congestion. The transition from free flow to congested flow is understood as a kind of phase transition. The OV model first reveals the transition mechanism very simply among car following models. Moreover, the model well reproduces the observed flow-density relation, so-called the fundamental diagram [13]. We recognize the OV model as a basic model for studying the phenomena of traffic flow.

In the OV model a driver is supposed to look at the preceding vehicle only. The reaction to the preceding vehicle plays an essential role to organize traffic congestion, and to explain the behavior of traffic flow. In more realistic situation, a driver looks at more vehicles around him, and the effect modifies the model. In the viewpoint of control theory for traffic flow, such a effect is important to suppress the formation of congestion. In this context, there have been several works to extend the OV model. In our previous papers [14,15], we discussed the improvement of stability when a driver looks at the vehicle that follows. Hayakawa and Nakanishi proposed another model for traffic and granular flow, which incorporates the effect of the particle that follows [16]. Nagatani proposed a model that a driver looks at the next to the preceding vehicle as well as the preceding vehicle [17]. The same model was also discussed by Sawada [18]. Lenz, Wagner, and Sollacher discussed a model that a driver looks at many vehicles ahead of him [19].

Automatic driving control systems are utilized as a part of so-called intelligent transport system (ITS). The suppression of traffic congestion is one of the targets of ITS. A cooperative driving control is one of such systems, where each vehicle receives information of many other vehicles and decides the optimal behavior. This system is expected to suppress the appearance of traffic congestion efficiently. In this paper we propose an extended OV model, in which a vehicle is controlled by the system using the information of arbitrary number of vehicles that precede or follow. We also discuss how this extension improves the stability of traffic flow. The extended model includes the above models [14–19] as special cases.

In Sec. II we present the extended OV model. We analyze the linear stability of uniform flow for the extended model in Sec. III. Section IV is devoted to the investigation for linear response of vehicles to a disturbance. Using results in this section, we present a dynamical model to control the real traffic. We summarize and discuss the whole results in Sec. V.

II. EXTENDED MODEL

The OV model is formulated by the following equation of motion:

$$\frac{d^2x_n}{dt^2}=a\left[V(\Delta x_n)-\frac{dx_n}{dt}\right], \tag{1}$$

where x_n and $\Delta x_n \equiv x_{n+1}-x_n$ are the position and the headway of nth vehicle, respectively. Vehicles are numbered such that the $(n+1)$th vehicle precedes the nth vehicle. We have introduced the OV function $V(\Delta x)$, which represents an optimal velocity of the vehicle with headway Δx. A driver controls the acceleration to decrease the difference between the optimal velocity and the real velocity. Parameter a, which has the dimension of inverse of time, is called sensitivity.

We extend OV model (1) to

*Email address: hasebe@vega.aichi-u.ac.jp
†Email address: g44153g@cc.nagoya-u.ac.jp
‡Email address: genbey@eken.phys.nagoya-u.ac.jp

$$\frac{d^2x_n}{dt^2}=a\left[V(\Delta x_{n+k_+},\ldots,\Delta x_{n+1},\Delta x_n,\Delta x_{n-1},\ldots,\Delta x_{n-k_-})-\frac{dx_n}{dt}\right]. \tag{2}$$

The OV function is extended to a function of $k_+ + k_- + 1$ variables, where $\Delta x_{n+k_+},\ldots,\Delta x_n$ are headways of the vehicles ahead of the nth vehicle, and $\Delta x_{n-1},\ldots,\Delta x_{n-k_-}$ are headways of the vehicles that follow. These variables are defined by $\Delta x_{n+k} \equiv x_{n+k+1} - x_{n+k}$ for $k=k_+, k_+-1,\ldots,-k_-$. The model with $k_+ = k_- = 0$ is the original OV model.

OV model (1) has a uniform flow solution

$$x_n = bn + V(b)t + \text{const}, \tag{3}$$

where all the vehicles have the same headway b and the same velocity $V(b)$. Extended model (2) has also a solution of uniform flow:

$$x_n = bn + V(b,b,\ldots,b)t + \text{const}. \tag{4}$$

We compare the properties of the extended model under the condition that the model has the same uniform flow solution as that of the original OV model. This condition imposes

$$V(b,\ldots,b) = V(b) \tag{5}$$

on the OV function for any extended models discussed in this paper. Under condition (5) we investigate the linear stability of the uniform flow and the linear response to a disturbance on the uniform flow of extended model for various k_+ and k_-.

III. LINEAR ANALYSIS OF EXTENDED MODEL

In this section we discuss the linear stability of the uniform flow. Let y_n be a small fluctuation imposed on the uniform flow. We assume the periodic boundary condition $x_{N+1} \equiv x_1$, where N is the total number of vehicles. From Eq. (2), $y_n(t)$ satisfies the linearized equation

$$\ddot{y}_n = a\left[\sum_{k=-k_-}^{k=k_+} f_k \Delta y_{n+k} - \dot{y}_n\right], \tag{6}$$

where $\Delta y_{n+k} = y_{n+k+1} - y_{n+k}$ and f_k is defined by

$$f_k = \frac{\partial}{\partial \Delta y_{n+k}} V(b+\Delta y_{n+k_+},\ldots,b+\Delta y_n,\ldots,b+\Delta y_{n-k_-})\big|_{\Delta y=0}, \quad -k_- \leqslant k \leqslant k_+. \tag{7}$$

We should choose the OV function such that $f_k > 0$ for $k \geqslant 0$ and $f_k < 0$ for $k < 0$. This choice is natural because of the following reasons. The positive value of $f_k (k \geqslant 0)$ has the effect of decreasing the velocity of nth vehicle if each headway of the vehicles that precede becomes small. On the other hand, the negative f_k $(k<0)$ has the effect of increasing the velocity if each headway of the vehicles that follow becomes small. Condition (5) with Eq. (7) becomes the condition for f_k as

$$\sum_{k=-k_-}^{k=k_+} f_k = V'(b). \tag{8}$$

By changing the scale for t and a, we can choose $V'(b) = 1$ without loss of generality.

We investigate the stability of mode solution $y_n(t) = \exp[in\theta - i\omega(\theta)t]$ of Eq. (6). In the same way as Ref. [12], we obtain the stability condition as

$$a > \frac{\left(\sum_k f_k\{\sin(k\theta) - \sin[(k+1)\theta]\}\right)^2}{\sum_k f_k\{\cos(k\theta) - \cos[(k+1)\theta]\}}. \tag{9}$$

The similar condition for an extended OV model is obtained in Ref. [19], which is a special case of our formula.

The criterion of stability is graphically understood in Fig. 1. We define $a(\theta, f_k)$ by the right-hand side of Eq. (9) as a function of θ for a given set of parameters $\{f_k | -k_- \leqslant k \leqslant k_+\}$. The solid curve in Fig. 1 shows the plot of $a(\theta, f_k)$ for a given $\{f_k\}$, in the polar coordinate (a,θ). The points corresponding to the mode solutions labeled by θ are distributed on this curve. For a given sensitivity a, we can draw the circle of radius a. In Fig. 1, we set $\{f_2, f_1, f_0\} = \{0.25, 0.25, 0.5\}$ and $a = 0.5$, for example. If the curve $a(\theta, f_k)$ crosses the circle, the corresponding modes on the part of the curve outside the circle make the uniform flow unstable. Thus, the uniform flow is stable for the case that

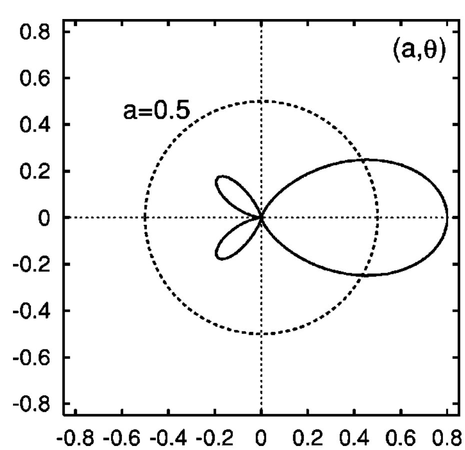

FIG. 1. The solid curve shows $a(\theta, f_k)$ in a polar coordinate (a,θ) for a set of parameters $\{f_2, f_1, f_0\} = \{0.25, 0.25, 0.5\}$. The dashed curve shows a circle of radius $a = 0.5$.

the curve $a(\theta,f_k)$ is completely inside the circle with radius a, i.e., $a>a(\theta,f_k)$ for all θ. In other words, the stability condition can be expressed as

$$a>\max_{\theta} a(\theta,f_k). \tag{10}$$

Here we consider a problem to find the minimum value of right-hand side of inequality (10) for a various choice of a set of parameters $\{f_k\}$, that is, to find the extended OV function $V(\ldots,\Delta x,\ldots)$, which makes the uniform flow "most stable." Then we consider a minimax problem

$$a_c \equiv \min_{f_k}\{\max_{\theta} a(\theta,f_k)\}, \tag{11}$$

under condition (8). We call a_c a critical sensitivity, and call the solution $\{f_k\}$ of Eq. (11) for given k_- and k_+ a set of "the most stable parameters."

First we consider an extended model with $k_-=0$ and $k_+>1$, which we call "forward looking" optimal velocity (FL-OV) models. In the models, a driver looks at vehicles in the direction of the vehicle. We perform a numerical search for a solution of the minimax problem for FL-OV models. In the result, we obtained

$$f_k=\frac{1}{k_++1}, \quad k=0,1,2,\ldots,k_+, \tag{12}$$

as a solution of Eq. (11) and the critical sensitivity as

$$a_c=\frac{2}{k_++1}. \tag{13}$$

The smallness of a_c compared to the value $a_c=2$ in the original OV model[1] shows the improvement of the stability in FL-OV models.

Let us illustrate the stability of a set of the most stable parameters. Solution (12) means

$$a(\theta,f_k)=\frac{1+\cos[(k_++1)\theta]}{k_++1}. \tag{14}$$

The maximum value of Eq. (14) is given by $\theta=2\pi m/(k_++1)$, $m=0,1,2,\ldots,k_+$, which is (k_++1)-fold degenerated. Figure 2 shows $a(\theta,f_k)$ for solution (14) in the case of $k_+=2$ together with another set of f_k, for the comparison. The curve $a(\theta,f_k)$ for $\{f_2,f_1,f_0\}=\{1/3,1/3,1/3\}$ looks like symmetric three leaves, which touch the circle $a_c=2/3$. The change of parameters from $\{1/3,1/3,1/3\}$ to $\{0.3,0.3,0.4\}$ causes two leaves to shrink and one leaf to spread, and then the corresponding critical sensitivity a_c is larger than 2/3. The behavior indicates the reason why $\{1/3,1/3,1/3\}$ gives the set of the most stable parameters.

Next, we consider another type of model, "backward-looking" optimal velocity (BL-OV) model, which is defined by $k_+=0$ and $k_-\geqslant 1$. In the model, a driver looks at just one preceding vehicle and the vehicles that follow. We choose the simplest case, $k_+=0$ and $k_-=1$, for example. By substituting the condition $f_0+f_{-1}=1$ into the right-hand side of Eq. (9), $a(\theta,f_k)$ is written as

$$a(\theta,f_0,f_{-1})=\frac{1}{2f_0-1}(1+\cos\theta). \tag{15}$$

Note that f_0 can take any large positive value because f_{-1} is negative. Then the minimax problem (11) has trivial solution $a_c=0$ at $f_0=\infty$. The situation is not changed for general BL-OV models. We emphasize that the stability in the BL-OV model presents the different aspect from that in the FL-OV model.

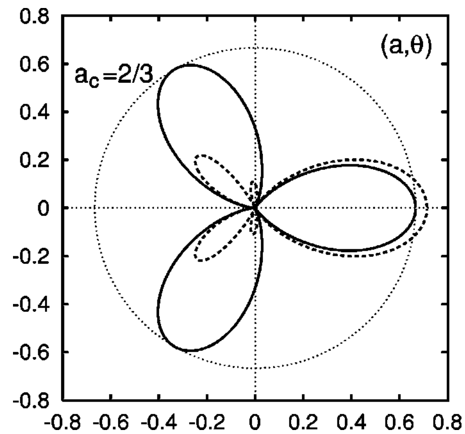

FIG. 2. Two examples of $a(\theta,f_k)$ are shown. The solid curve shows $a(\theta,f_k)$ for the set of the most stable parameters $\{f_2,f_1,f_0\}=\{1/3,1/3,1/3\}$, in a polar coordinate (a,θ). The dashed curve shows another $a(\theta,f_k)$ for $\{0.3,0.3,0.4\}$. The dotted curve shows a circle of radius $a_c=2/3$.

IV. LINEAR RESPONSE

In the preceding section, we analyzed the stability condition of the uniform flow solution in extended models. Here, we discuss the linear response to the disturbance imposed on the uniform flow. For the purpose of measuring the dynamical behavior for the stability of the uniform flow, we introduce two test functions

$$A(t)=\frac{1}{\epsilon^2}\sum_n y_n^2(t), \tag{16}$$

$$B(t)=\frac{1}{\epsilon^2}\sum_n \dot{y}_n^2(t), \tag{17}$$

where $y_n(t)$ and $\dot{y}_n(t)$ are the fluctuations of the position and the velocity of the nth vehicle. The uniform flow is disturbed at $t=0$ such that only one vehicle changes its position x to $x+\epsilon$ without changing velocity,

$$y_n(0)=\epsilon\delta_{n0}, \quad \dot{y}_n(0)=0. \tag{18}$$

[1]Before changing the time scale, such that $V'(b)=1$, the critical sensitivity is $a_c=2V'(b)$ [12].

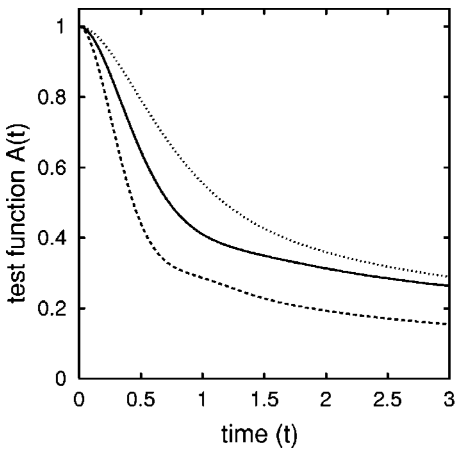

FIG. 3. The behavior of the test function $A(t)$ with $a=3$: the dashed curve, the solid curve, and the dotted curve represent $A(t)$ for the BL-OV model, the original OV model, and the FL-OV model, respectively.

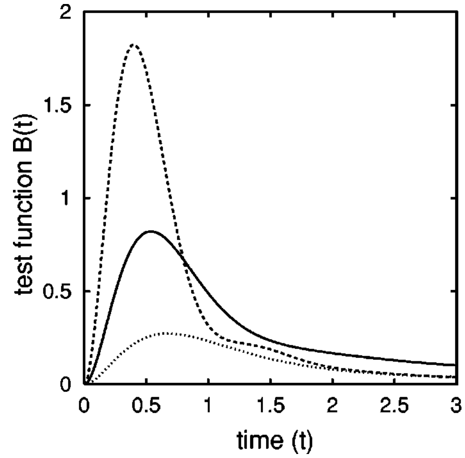

FIG. 4. The behavior of the test function $B(t)$ with $a=3$: the dashed curve, the solid curve, and the dotted curve represent $B(t)$ for the BL-OV model, the original OV model and the FL-OV model, respectively.

The solution of Eq. (6) in this initial condition is written as

$$y_n(t)=\frac{\epsilon}{N}\sum_{\theta,\sigma}\frac{\omega_{-\sigma}(\theta)}{\omega_{-\sigma}(\theta)-\omega_{\sigma}(\theta)}\exp[in\theta-i\omega_{\sigma}(\theta)t], \tag{19}$$

where $\sigma=\pm$ is an index for two mode solutions of Eq. (6).

Now we discuss the linear responses for three models: the original OV model, the BL-OV model with $k_-=1$, and the FL-OV model with $k_+=1$. We set the sensitivity parameter $a=3$, in which value the uniform flow is stable for the above three models. The parameters f_k are set as $\{f_0,f_{-1}\}=\{1.5,-0.5\}$ for the BL-OV model, $f_0=1$ for the original OV model and $\{f_1,f_0\}=\{0.5,0.5\}$ for the FL-OV model, which is the set of the most stable parameters. In Fig. 3, the behaviors of $A(t)$, which measures the fluctuation of position, are presented for the three models. In the BL-OV model, the disturbance damps faster than the other models, which is emphasized in our previous paper [15]. The disturbance slowly damps in the FL-OV model. However, this result does not necessarily mean the inferiority of the FL-OV model. In fact, the FL-OV model shows a small amount of fluctuation for the test function $B(t)$, which measures the fluctuation of velocity (see Fig. 4). In contrast, the BL-OV model shows a large amount of the fluctuation of velocity. This is just the opposite result for the behavior of the test function $A(t)$.

In Fig. 5, we show the damping behavior of test function $A(t)$ in the BL-OV model for various parameters compared with the original OV model. The sets of parameters $\{f_k\}$ are as follows: $\{f_0,f_{-1}\}=\{1.1,-0.1\}$, $\{1.3,-0.3\}$, $\{1.5,-0.5\}$. The disturbance damps faster as the parameter f_0 becomes larger under the condition $f_0+f_{-1}=1$. This result is expected from the stability condition derived by Eq. (15).

In Fig. 6, we show the amplitude of velocity-fluctuation $B(t)$ in the FL-OV models for various $\{f_k\}$ compared with the original OV model. The sets of parameters are as follows: $\{f_1,f_0\}=\{0.1,0.9\}$, $\{0.3,0.7\}$, $\{0.5,0.5\}$. The amplitude becomes smaller as the set of parameters approaches that of the most stable parameters (12). This result is expected from the result of the preceding section.

Next we discuss more complicated models. One is the BL-OV model of $k_-=2$, in which a driver looks at the preceding vehicle and two successive vehicles that follow. Another is the FL-OV model of $k_+=2$, in which a driver looks at the preceding vehicle and the next two successive vehicles to the preceding vehicle. The other is a model of $k_-=1$, $k_+=1$, which we call a hybrid OV (HB-OV) model. In this model, a driver looks at the preceding vehicles, the next to the preceding vehicle and the vehicle that follows. The set of parameters $\{f_k\}$ for each model is listed in Table I.

Figure 7 represents the damping behavior of the test function $A(t)$ for the above three models. The sensitivity parameter a is set to 3. In the HB-OV model, the behavior of $A(t)$ is almost the same as the BL-OV model. This indicates that the property of fluctuation of position in the BL-OV model is

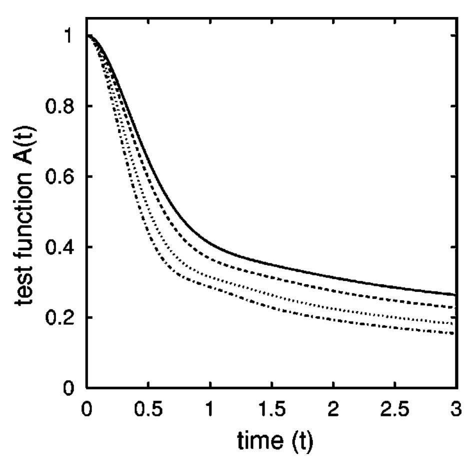

FIG. 5. The solid curve represents the behavior of the test function $A(t)$ for the original OV model. The dashed, dotted, and dash-dotted curves represent $A(t)$ for the BL-OV models, which have the parameters $\{f_0,f_{-1}\}=\{1.1,-0.1\}$, $\{1.3,-0.3\}$, $\{1.5,-0.5\}$, respectively.

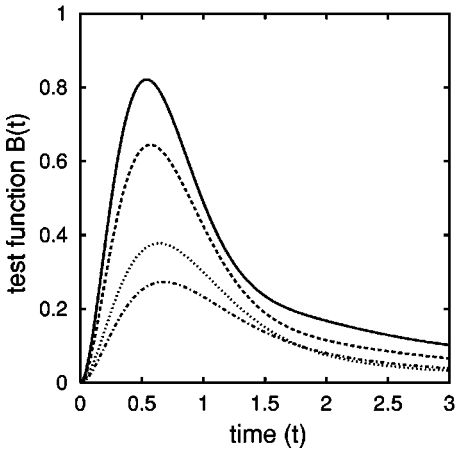

FIG. 6. The solid curve represents the behavior of the test function $B(t)$ for the original OV model. The dashed, dotted, and dash-dotted curves represent $B(t)$ for the FL-OV models, which have the parameters $\{f_1,f_0\}=\{0.1,0.9\}$, $\{0.3,0.7\}$, $\{0.5,0.5\}$, respectively.

not changed so strongly by introducing ''forward-looking'' effect. By comparing Figs. 3 and 7, we find that $A(t)$ in the BL-OV model of $k_-=2$ damps faster than that in the model of $k_-=1$. In the FL-OV model of $k_+=2$, $A(t)$ damps slower than that in the model of $k_+=1$. This behavior of FL-OV models will be explained in detail in the following section.

Figure 8 represents the damping behavior of the test function $B(t)$. In the FL-OV model of $k_+=2$, the amplitude of fluctuation of velocity $B(t)$ becomes smaller than that in the FL-OV model of $k_+=1$ (see Fig. 4). In the BL-OV model of $k_-=2$, the amplitude of $B(t)$ becomes larger than that in the BL-OV model of $k_-=1$. This means that vehicles respond sensitively to the initial disturbance in the BL-OV model. In the HB-OV model, the behavior of $B(t)$ is almost the same as the original OV model. The HB-OV model is a kind of BL-OV model including the ''forward-looking'' effect. We can understand that the property of fluctuation of velocity in the BL-OV model is improved by introducing ''forward-looking'' effect.

In the FL-OV model, the fluctuation of velocity is small and the fluctuation of position damps slowly. In other words, the FL-OV model controls the motion of vehicle ''mildly.'' The BL-OV model has property just opposite to the FL-OV model. The fluctuation of velocity is large and the fluctuation of position damps fast. The BL-OV model controls a vehicle ''severely.'' The above two models are complementary in the property of the response to the disturbance. In the HB-OV model, the fluctuation of velocity is suppressed compared to

TABLE I. The sets of parameters $\{f_k\}$ for the BL-OV, HB-OV, and FL-OV models.

Model	k_+	k_-	f_2	f_1	f_0	f_{-1}	f_{-2}
BL-OV	0	2			2.0	-0.5	-0.5
HB-OV	1	1		0.5	1.0	-0.5	
FL-OV	2	0	1/3	1/3	1/3		

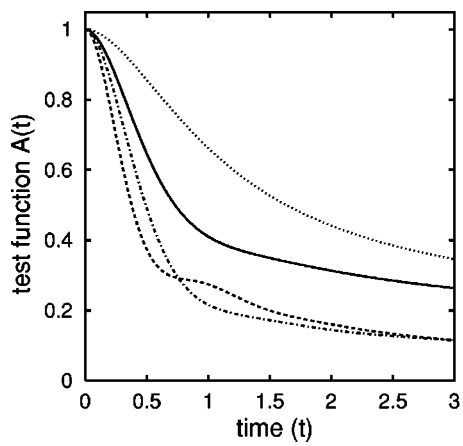

FIG. 7. The solid curve represents the behavior of the test function $A(t)$ for the original OV model. The dashed, dotted, and dash-dotted curves represent $A(t)$ for the BL-OV model, the FL-OV model, and the HB-OV model listed in Table I.

the BL-OV model, and moreover the fluctuation of position damps as fast as the BL-OV model. We insist that the HB-OV model is a good candidate to control real traffic flow on a highway.

V. SUMMARY AND DISCUSSION

In this paper we discussed an extended OV model for the purpose of constructing of a driving system for freeway traffic. In the model, a vehicle is controlled by the system using the information of arbitrary number of vehicles that precede or follow. The properties of the model are compared under the condition that it has the same uniform flow solution as that of the original OV model for any value of headway. We investigated the dynamical properties for the stability by calculating the response to the disturbance imposed on the uniform flow. In the FL-OV model we obtained the set of most stable parameters, which gives the highest stability, whereas the idea of such kind of parameter is useless for the BL-OV model. The FL-OV model and the BL-OV model are

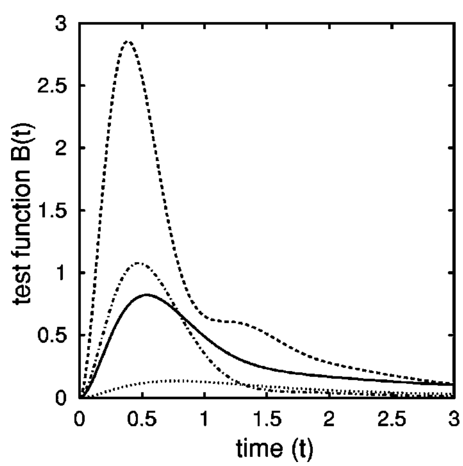

FIG. 8. The solid curve represents the behavior of the test function $B(t)$ for the original OV model. The dashed, dotted, and dash-dotted curves represent $B(t)$ for the BL-OV model, the FL-OV model, and the HB-OV model listed in Table I.

complementary in the property of response. In the FL-OV model, the fluctuation of velocity is small and the fluctuation of position damps slowly. On the other hand, in the BL-OV model the fluctuation of velocity is large and the fluctuation of position damps fast. On the basis of these analysis, we have shown that the HB-OV model inherits the superior properties from both models. The HB-OV model is a candidate as a dynamical model of cooperative driving system that controls real traffic flow on a highway. The dynamical effects of looking at the vehicles that proceed or follow investigated in this paper could be common to other types of models, and our results offer the important information for constructing a control theory for traffic flow.

ACKNOWLEDGMENT

This work was partly supported by a Grant-in-Aid for Scientific Research (C) (No. 13640409) of the Japanese Ministry of Education, Science, Sports and Culture.

[1] *Traffic and Granular Flow*, edited by D.E. Wolf and M. Schreckenberg, and A. Bachem (World Scientific, Singapore, 1996).
[2] *Traffic and Granular Flow '97*, edited by M. Schreckenberg and D.E. Wolf (Springer-Verlag, Singapore, 1998).
[3] *Traffic and Granular Flow '99*, edited by D. Helbing, H.J. Herrmann, M. Schreckenberg, and D.E. Wolf (Springer-Verlag, Berlin, 2000).
[4] D. Chowdhury, Ludger Santen, and A. Schadschneider, Phys. Rep. **329**, 199 (2000).
[5] D. Helbing, Rev. Mod. Phys. **73**, 1067 (2001).
[6] K. Nagel and M. Schreckenberg, J. Phys. I **2**, 2221 (1992).
[7] T. Nagatani, J. Phys. Soc. Jpn. **62**, 3837 (1993); Phys. Rev. E **48**, 3290 (1993).
[8] M. Fukui and Y. Ishibashi, J. Phys. Soc. Jpn. **62**, 3841 (1993).
[9] B.S. Kerner and P. Konhäuser, Phys. Rev. E **48**, 2335 (1993).
[10] S. Yukawa, M. Kikuchi, and S. Tadaki, J. Phys. Soc. Jpn. **63**, 3609 (1994); S. Yukawa and M. Kikuchi, *ibid.* **64**, 35 (1995).
[11] R. Mahnke and N. Pieret, Phys. Rev. E **56**, 2666 (1997).
[12] M. Bando, K. Hasebe, A. Nakayama, A. Shibata, and Y. Sugiyama, Phys. Rev. E **51**, 1035 (1995); Jpn. J. Indust. Appl. Math. **11**, 203 (1994).
[13] M. Bando, K. Hasebe, K. Nakanishi, A. Nakayama, A. Shibata, and Y. Sugiyama, J. Phys. I **5**, 1389 (1995).
[14] K. Hasebe, in *Proceedings of the 3rd Symposium on Simulation of Traffic Flow*, edited by M. Fukui and Y. Ishibashi (The Mathematical Society of Traffic Flow, Japan, 1997), p. 6 (in Japanese).
[15] A. Nakayama, Y. Sugiyama, and K. Hasebe, Phys. Rev. E **65**, 016112 (2002).
[16] H. Hayakawa and K. Nakanishi, Prog. Theor. Phys. Suppl. **130**, 57 (1998).
[17] T. Nagatani, Phys. Rev. E **60**, 6395 (1999).
[18] S. Sawada, Int. J. Mod. Phys. C **13**, 1 (2002).
[19] H. Lenz, C.K. Wagner, and R. Sollacher, Eur. Phys. J. B **7**, 331 (1999).